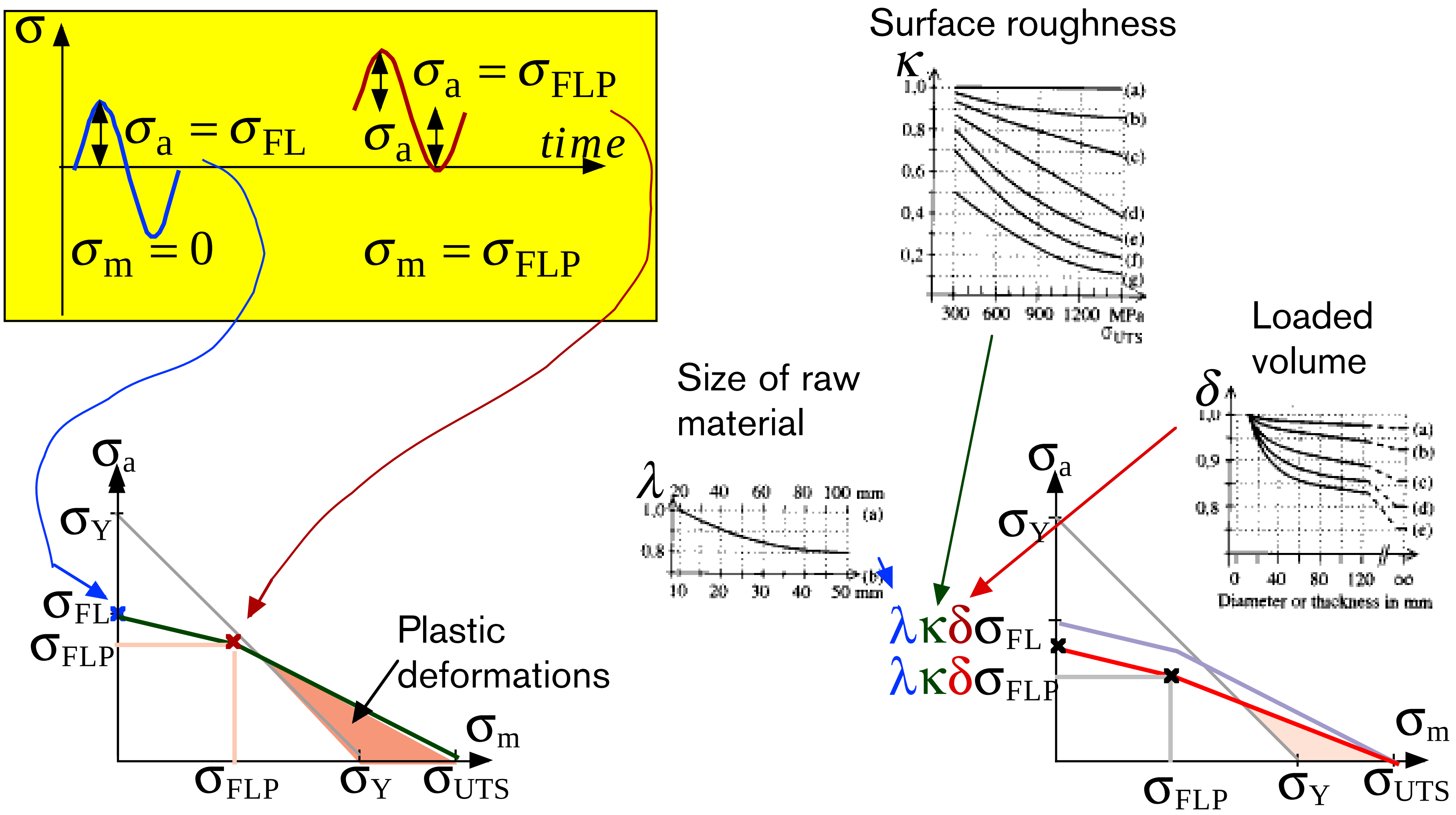
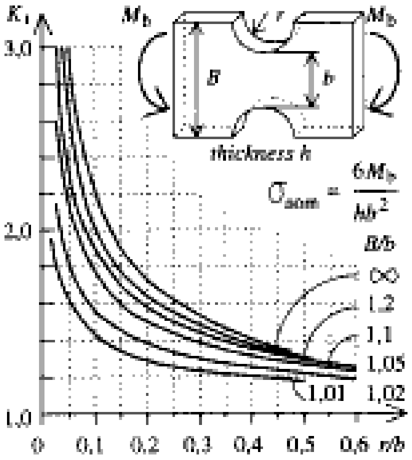
Haigh diagram I



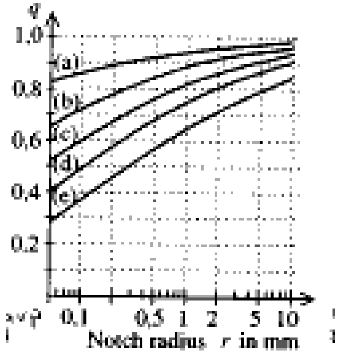
Haigh diagram

Reduced Haigh diagram

Haigh diagram II



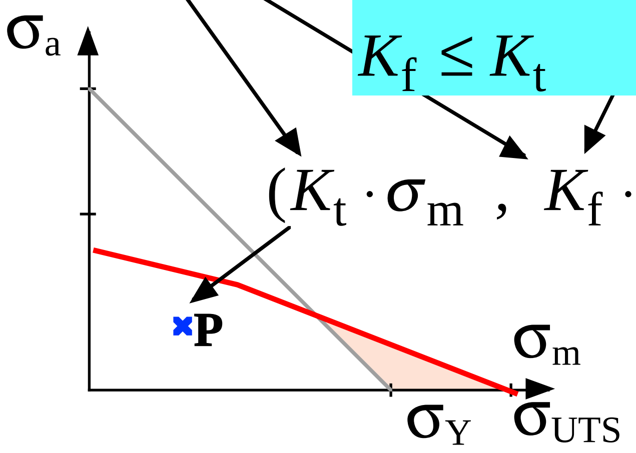
Bending of flat bar with notch



$$K_f = 1 + q(K_t - 1)$$

$$K_f \le K_t$$

$$(K_t \cdot \sigma_m \ , \ K_f \cdot \sigma_a)$$



Service stress

$$SF_a = \frac{AA'}{AP}$$

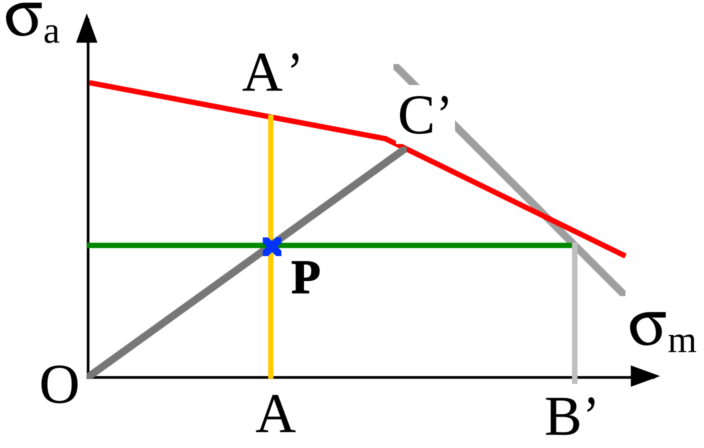
$$\sigma_m = \text{const}$$

$$SF_m = \frac{OB'}{OA}$$

$$\sigma_a = \text{const}$$

$$SF_{am} = \frac{OC'}{OP}$$

$$\frac{K_f \cdot \sigma_a}{K_t \cdot \sigma_m} = \text{const}$$



Safety factors

“Modern” Fatigue Design

Background

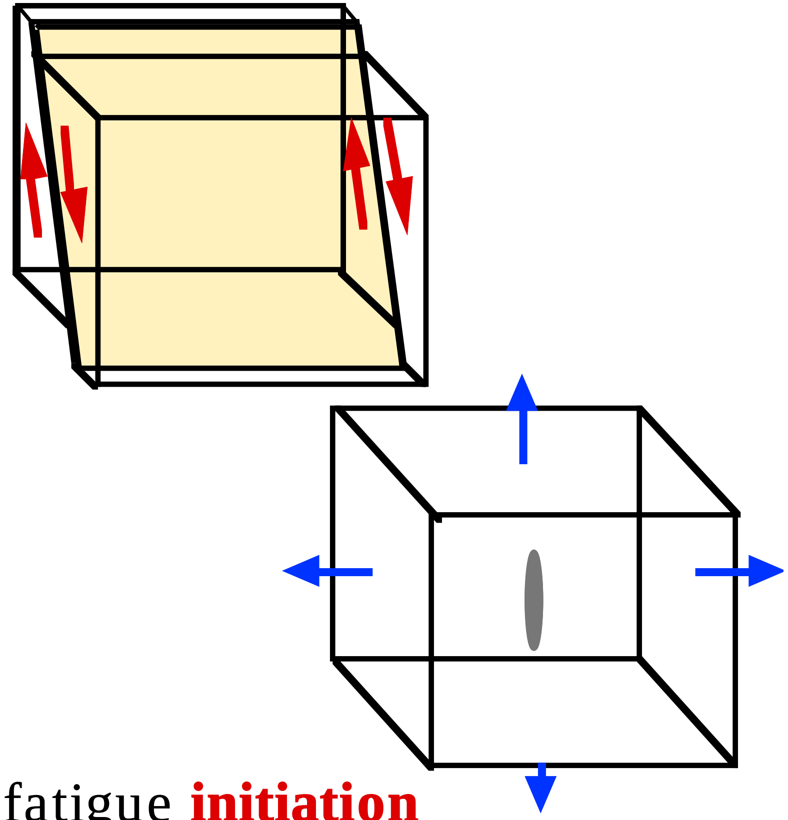
◇ Evolution in structural design due to
  - increased computational power
  - CAD/CAE - software

◇ Need for new fatigue design methods that are
  - valid for a **general type of loading**
  - **easy to implement** in a **computer code**

◇ Several options, but **no method with general validity**

- **HCF**: **equivalent stress** is defined and compared to a **fatigue limit** (expressed in the equivalent stress)
- **LCF**: calculation of damage connected to the **constitutive model** of the material. Fatigue damage connected to the **plastic deformation**
- **LEFM**: effective stress intensity range

Multiaxial high cycle fatigue initiation



Problem:

- The Haigh diagram is valid for
  - Uniaxial loading
  - One stress component

Solution:

- Assume that, in the general case, fatigue behaviour is influenced by
  - Applied **shear stress amplitude**
  - **Hydrostatic stress**

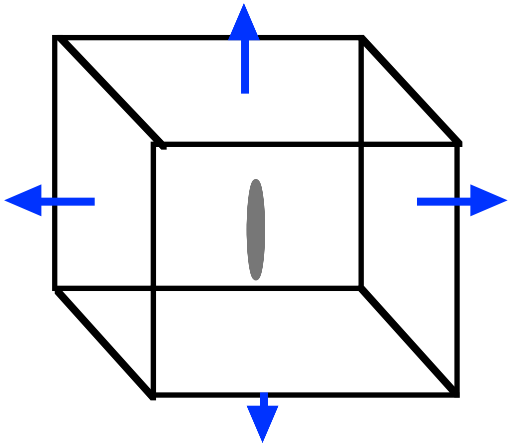
Based on these assumptions, derive a fatigue **initiation** criterion that defines a **limiting stress magnitude** for which fatigue cracks will develop) for a general type of loading.

Assumes undamaged material (continuum mechanics)

Hydrostatic stress

- The hydrostatic stress is the **mean value** of **normal stresses** acting on the material point (positive in tension)
- A tensile (positive) hydrostatic stress opens up microscopic cracks (**Stage II** crack growth)

$$\sigma_\mathrm{h} = \frac{1}{3}\left(\sigma_x + \sigma_y + \sigma_z\right)$$



- The hydrostatic stress is a stress invariant

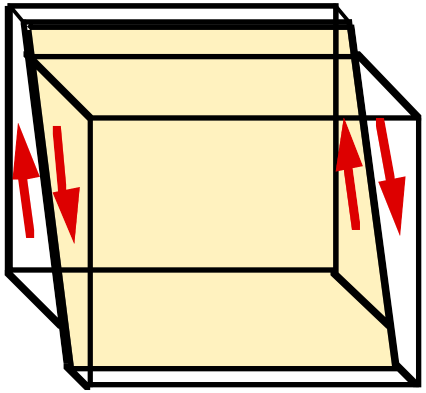
$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$$\sigma_\mathrm{h} = \frac{1}{3}\sigma_{ii} = \frac{1}{3}\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)$$

regardless of coordinate system

Shear stress measures

- The shear stress initiates slip bands which leads to microscopic cracks (**stage I** crack growth)
- Since **a static shear stress have no influence on the fatigue damage**, the shear stress ”*amplitude*” is employed



- Two measures
  - Tresca shear stress

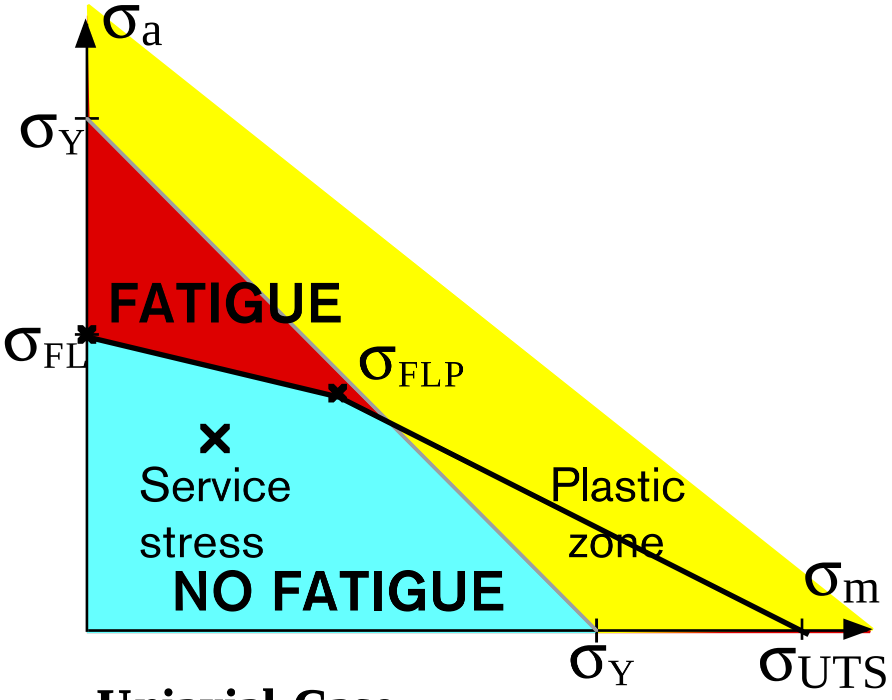
$$\tau_{\text{Tresca}} = \frac{\sigma_1 - \sigma_3}{2}$$

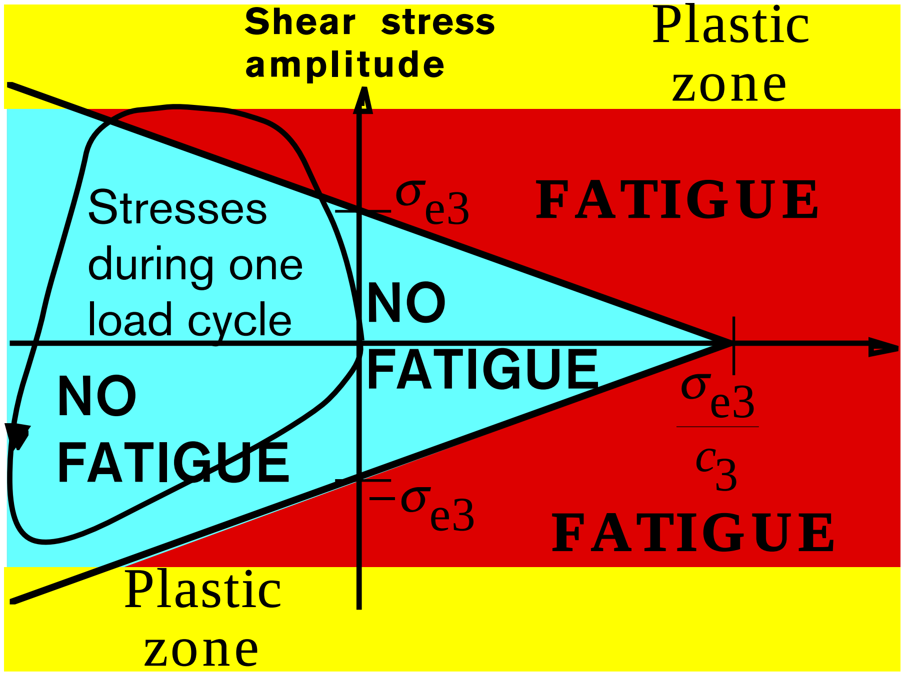
  - von Mises stress

$$\sigma_{\text{vM}} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

We need to define the “*amplitudes*” of these

Equivalent Stress Measures





Uniaxial Case

- One stress component
- **Mid value** and **amplitude** of this stress component are taken to reflect the fatigue properties
- The stresses during a load cycles are defined by a service stress

Multiaxial Case

- Six stress components (general case)
- **Hydrostatic** stress and **shear** stress “amplitude” are taken to reflect the fatigue properties
- The stresses during a load cycles are defined by a closed curve

Shear Stress “Amplitude”

General

- It has been found empirically that a **superposed static shear stress** does not have any influence on the fatigue initiation

  $\tau_{\mathrm{FL}} = \tau_{\mathrm{FLP}}$ whereas $\sigma_{\mathrm{FL}} \neq \sigma_{\mathrm{FLP}}$

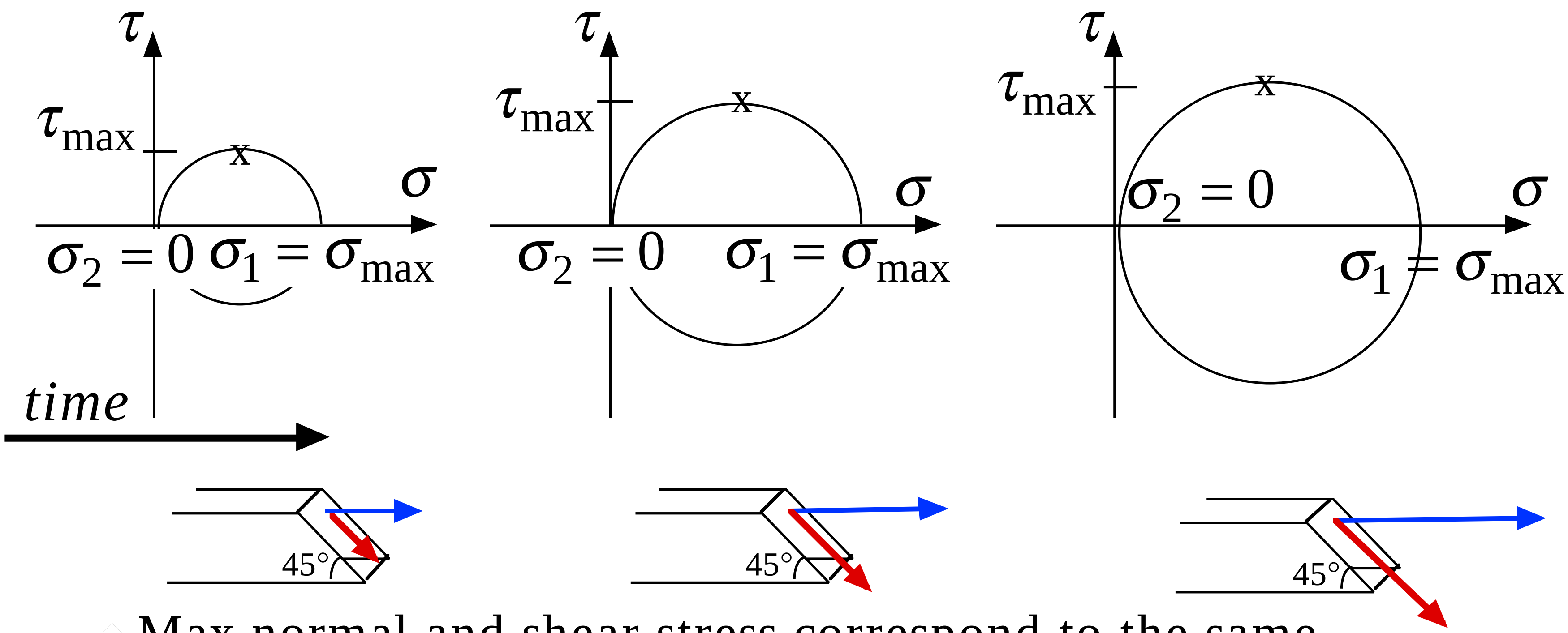
- In order to eliminate the influence of a superposed shear stress, the shear stress **”amplitude”** is normally used in multiaxial HCF-criteria

- This ”*amplitude*” is the **difference between** the **current shear stress** magnitude and the **mid value of the shear stress** for the **current stress cycle**

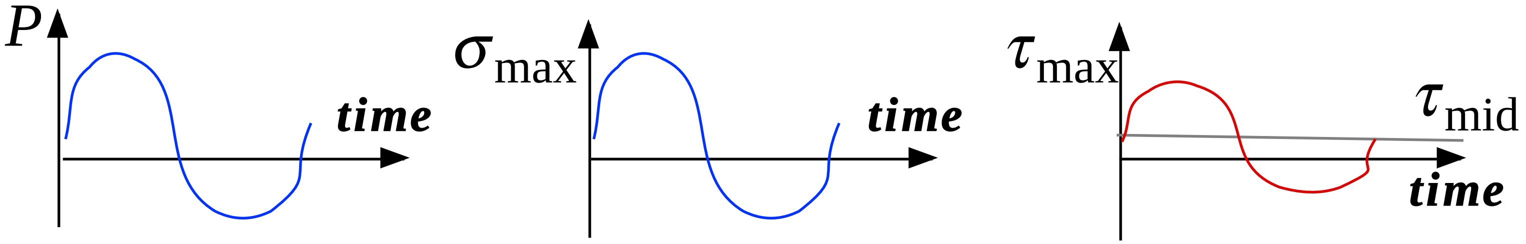
- For the general case, this “*amplitude*” is rather complicated to compute (see Fatigue – a Survey, Appendix I)

Shear stress – Uniaxial case

- Mohr's stress circle for loading in a uniaxial case



- Max normal and shear stress correspond to the same directions throughout the load cycle



The deviatoric stress tensor

The stress tensor

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

Split into **volumetric** and a **deviatoric** part

$$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} = \sigma_{\mathrm{h}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \sigma_{xx} - \sigma_{\mathrm{h}} & \tau_{xy} & \tau_{xz} \\ \tau_{yz} & \sigma_{yy} - \sigma_{\mathrm{h}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} - \sigma_{\mathrm{h}} \end{bmatrix}$$

$$= \sigma_{\mathrm{h}} \mathbf{I} + \boldsymbol{\sigma}^{\mathrm{d}}$$

The **volumetric** part contains the **hydrostatic stress**

The **deviatoric** part reflects influence of **shear stresses**

Mid value of the deviatoric stress tensor

In-phase

$$\sigma_{ij} = a_{ij} + c_{ij} \cdot f(t)$$

- a_{ij} and c_{ij} are constants
- $f(t)$ is a common time dependent function

Fixed principal directions

Every component of corresponds to a fixed direction throughout the loading

$$\sigma_{ij}^{\mathrm{d}}(t) = \begin{bmatrix} \sigma_{xx}^{\mathrm{d}}(t) & \tau_{xy}^{\mathrm{d}}(t) & \tau_{xz}^{\mathrm{d}}(t) \\ \tau_{yx}^{\mathrm{d}}(t) & \sigma_{yy}^{\mathrm{d}}(t) & \tau_{yz}^{\mathrm{d}}(t) \\ \tau_{zx}^{\mathrm{d}}(t) & \tau_{zy}^{\mathrm{d}}(t) & \sigma_{zz}^{\mathrm{d}}(t) \end{bmatrix} = \begin{bmatrix} \sigma_1^{\mathrm{d}}(t) & 0 & 0 \\ 0 & \sigma_2^{\mathrm{d}}(t) & 0 \\ 0 & 0 & \sigma_3^{\mathrm{d}}(t) \end{bmatrix}$$

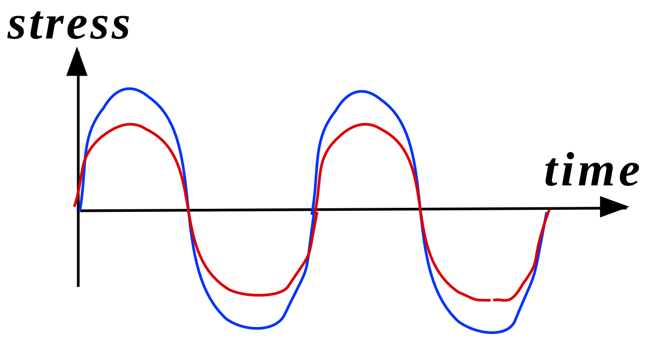
$$\Rightarrow$$

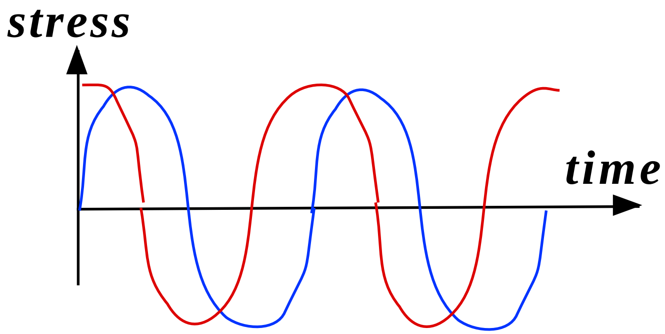
$$= \begin{bmatrix} a_{11}^{\mathrm{d}} & 0 & 0 \\ 0 & a_{22}^{\mathrm{d}} & 0 \\ 0 & 0 & a_{33}^{\mathrm{d}} \end{bmatrix} + \begin{bmatrix} c_{11}^{\mathrm{d}} & 0 & 0 \\ 0 & c_{22}^{\mathrm{d}} & 0 \\ 0 & 0 & c_{33}^{\mathrm{d}} \end{bmatrix} \cdot f(t)$$

Movie 1 – *Click me!*

Mid value of the deviatoric stress tensor III

Limitations – In-phase loading





- In **in-phase loading**, the stresscomponents have their max- and min-magnitudes at **the same instant in time**

- In **out-of-phase** loading, max- and min magnitudes occur at **different instants of time** for different stress components

- The case of out-of-phase loading is much more difficult to analyse, for instance due to difficulties in
  - Defining a stress cycle
  - Defining a mid value of the shear stress

Mid value of the deviatoric stress tensor IV

Limitations – Fixed principal directions

Rotating principal directions ⇒

$$"\sigma_{ij,\mathrm{p}}^{\mathrm{d}}" = \begin{bmatrix} \sigma_1^{\mathrm{d}} & 0 & 0 \\ 0 & \sigma_1^{\mathrm{d}} & 0 \\ 0 & 0 & \sigma_3^{\mathrm{d}} \end{bmatrix}$$

corresponds to a rotating coordinate system

Instead we have to look at the full deviatoric stress tensor and find its mid value

$$\sigma_{ij,\mathrm{m}}^{\mathrm{d}} = \boldsymbol{\sigma}_{\mathrm{m}}^{\mathrm{d}} = \begin{bmatrix} (\sigma_{xx} - \sigma_{\mathrm{h}}) & (\tau_{xy}) & (\tau_{xz}) \\ (\tau_{yz}) & (\sigma_{yy} - \sigma_{\mathrm{h}}) & (\tau_{yz}) \\ (\tau_{zx}) & (\tau_{zy}) & (\sigma_{zz} - \sigma_{\mathrm{h}}) \end{bmatrix}_{\mathrm{m}}$$

Mid value of the deviatoric stress tensor V

Finding the mid value in a general case – *Click me!*

“Amplitude” of the deviatoric stress tensor

The **mid value** of the deviatoric stress tensor is found as

$$\sigma^{\mathrm{d}}_{ij,\mathrm{m}} = \boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{m}} = \begin{bmatrix} \sigma^{\mathrm{d}}_{1\,\mathrm{m}} & 0 & 0 \\ 0 & \sigma^{\mathrm{d}}_{1\,\mathrm{m}} & 0 \\ 0 & 0 & \sigma^{\mathrm{d}}_{3\,\mathrm{m}} \end{bmatrix} \text{(proportional loading)}$$

(“m” denotes mid-value of component during stress cycle)

or as

$$\sigma^{\mathrm{d}}_{ij,\mathrm{m}} = \boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{m}} = \begin{bmatrix} (\sigma_{xx} - \sigma_{\mathrm{h}}) & (\tau_{xy}) & (\tau_{xz}) \\ (\tau_{yz}) & (\sigma_{yy} - \sigma_{\mathrm{h}}) & (\tau_{yz}) \\ (\tau_{zx}) & (\tau_{zy}) & (\sigma_{zz} - \sigma_{\mathrm{h}}) \end{bmatrix}_{\mathrm{m}} \text{(general)}$$

the “*amplitude*” of the deviatoric stress tensor is defined as

$$\sigma^{\mathrm{d}}_{ij,\mathrm{a}}(t) = \sigma^{\mathrm{d}}_{ij}(t) - \sigma^{\mathrm{d}}_{ij,\mathrm{m}} \text{ (or } \boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{a}}(t) = \boldsymbol{\sigma}^{\mathrm{d}}(t) - \boldsymbol{\sigma}^{\mathrm{d}}_{\mathrm{m}})$$

“Amplitude” of the deviatoric stress tensor II

For in-phase loading with fixed principal directions (proportional loading), we can express the “*amplitude*” of the Tresca and von Mises stress using the “*amplitude*” of the deviatoric stress tensor

$$\tau_{\text{Tresca,a}}(t) = \frac{\sigma^{\text{d}}_{1,\text{a}}(t) - \sigma^{\text{d}}_{3,\text{a}}(t)}{2} \quad \text{where } (\sigma^{\text{d}}_{1,\text{a}}(t) = \sigma^{\text{d}}_{1}(t) - \sigma^{\text{d}}_{1,\text{m}}\ etc)$$

$$\sigma_{\text{vM,a}}(t) = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{1,\text{a}}(t) - \sigma_{2,\text{a}}(t)\right)^2 + \left(\sigma_{2,\text{a}}(t) - \sigma_{3,\text{a}}(t)\right)^2 + \left(\sigma_{3,\text{a}}(t) - \sigma_{1,\text{a}}(t)\right)^2}$$

(it can be shown that using σ_{a} or $\sigma^{\text{d}}_{\text{a}}$ gives the same results)

The max values are given as

$$\tau_{\text{Tresca,a}} = \frac{\sigma^{\text{d}}_{1,\text{a}} - \sigma^{\text{d}}_{3,\text{a}}}{2} \quad \text{where } (\sigma^{\text{d}}_{1,\text{a}} = \frac{\sigma^{\text{d}}_{1,\text{max}} - \sigma^{\text{d}}_{1,\text{min}}}{2})$$

$$\sigma_{\text{vM,a}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{1,\text{a}} - \sigma_{2,\text{a}}\right)^2 + \left(\sigma_{2,\text{a}} - \sigma_{3,\text{a}}\right)^2 + \left(\sigma_{3,\text{a}} - \sigma_{1,\text{a}}\right)^2}$$

Equivalent stress criteria

Sines criterion

$$\sigma_{\mathrm{EQS}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{1,\mathrm{a}}^{\mathrm{d}} - \sigma_{2,\mathrm{a}}^{\mathrm{d}}\right)^2 + \left(\sigma_{2,\mathrm{a}}^{\mathrm{d}} - \sigma_{3,\mathrm{a}}^{\mathrm{d}}\right)^2 + \left(\sigma_{3,\mathrm{a}}^{\mathrm{d}} - \sigma_{1,\mathrm{a}}^{\mathrm{d}}\right)^2} + c_{\mathrm{S}}\sigma_{\mathrm{h,mid}} > \sigma_{\mathrm{eS}}$$

Crossland criterion

$$\sigma_{\mathrm{EQC}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{1,\mathrm{a}} - \sigma_{2,\mathrm{a}}\right)^2 + \left(\sigma_{2,\mathrm{a}} - \sigma_{3,\mathrm{a}}\right)^2 + \left(\sigma_{3,\mathrm{a}} - \sigma_{1,\mathrm{a}}\right)^2} + c_{\mathrm{C}}\sigma_{\mathrm{h,max}} > \sigma_{\mathrm{eC}}$$

Dang van criterion

$$\sigma_{\mathrm{EQDV}} = \frac{\sigma_{1,\mathrm{a}} - \sigma_{3,\mathrm{a}}}{2} + c_{\mathrm{DV}}\sigma_{\mathrm{h,max}} > \sigma_{\mathrm{eDV}}$$

Concluding remarks

Fatigue analysis

- Calculate the state of stress
- Apply the equivalent stress criterion, fatigue if

$$\sigma_{eq} > \sigma_e$$

- In the case of no fatigue, calculate safety coefficient as

$$SF = {}^{\sigma_e}\!/_{\sigma_{EQ}}$$

Pros	**Cons**
Suitable for computer analysis	Corrosion correction etc.
General state of stress	Lack of empirical knowledge
Identify critical parts of component	Separates between fatigue / no fatigue
Have a physical basis	

Lunch